

# Products of Beta distributed random variables

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**ABSTRACT.** This is an expository note on useful expressions for the density function of a product of independent random variables where each variable has a Beta distribution.

## 1. Introduction

This is a brief exposition of some techniques to construct density functions with moment sequences of the form  $\prod_{j=1}^m \frac{(u_j)_n}{(v_j)_n}$ , where  $(a)_n$  denotes the Pochhammer symbol  $\frac{\Gamma(a+n)}{\Gamma(a)}$ . Such a density  $f(x)$  can be expressed as a certain Meijer G-function, that is, a sum of generalized hypergeometric series, and as a power series in  $(1-x)$  whose coefficients can be calculated by a recurrence. The former expression is pertinent for numerical computations for  $x$  near zero, while the latter is useful for  $x$  near 1.

All the random variables considered here take values in  $[0, 1]$ , density functions are determined by their moments: for a random variable  $X$  we have  $P[X < a] = \int_0^a f(x) dx$  for  $0 \leq a \leq 1$ , and the expected value  $E(X^n) = \int_0^1 x^n f(x) dx$  is the  $n$ th moment. The basic building block is the Beta distribution  $(\alpha, \beta > 0)$

$$(1.1) \quad h(\alpha, \beta; x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1},$$

where  $B(\alpha, \beta) := \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ , then

$$\int_0^1 x^n h(\alpha, \beta; x) dx = \frac{(\alpha)_n}{(\alpha + \beta)_n}, n = 0, 1, 2, \dots,$$

thus  $\left\{ \frac{(u)_n}{(v)_n} \right\}$  is a moment sequence if  $0 < u < v$  (with  $\alpha = u, \beta = v - u$ ). The moments of the product of independent random variables are the products of the respective moments, that is, suppose the densities of (independent)  $X$  and  $Y$  are  $f, g$  respectively and define

$$(1.2) \quad f * g(x) = \int_x^1 f(t) g\left(\frac{x}{t}\right) \frac{dt}{t},$$

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then  $f * g$  is a density,  $P[XY < a] = \int_0^a f * g(x) dx$  for  $0 \leq a \leq 1$  and

$$\int_0^1 x^n (f * g(x)) dx = \int_0^1 x^n f(x) dx \int_0^1 y^n g(y) dy.$$

These are the main results: suppose the parameters  $u_1, \dots, u_m$  and  $v_1, \dots, v_m$  satisfy  $v_i > u_i > 0$  for each  $i$ , then there is a unique density function  $f$  with the moment sequence  $\prod_{j=1}^m \frac{(u_j)_n}{(v_j)_n}$ ;

(1) if also  $u_i - u_j \notin \mathbb{Z}$  for each  $i \neq j$  then for  $0 \leq x < 1$

$$(1.3) \quad f(x) = \left( \prod_{k=1}^m \frac{\Gamma(v_k)}{\Gamma(u_k)} \right) \sum_{i=1}^m \frac{1}{\Gamma(v_i - u_i)} \prod_{j=1, j \neq i}^m \frac{\Gamma(u_j - u_i)}{\Gamma(v_j - u_i)} x^{u_i - 1} \\ \times \sum_{n=0}^{\infty} \prod_{k=1}^m \frac{(u_i - v_k + 1)_n}{(u_i - u_k + 1)_n} x^n;$$

(2) for  $\delta := \sum_{i=1}^m (v_i - u_i)$  there is an  $(m+1)$ -term recurrence for the coefficients  $\{c_n\}$  such that

$$(1.4) \quad f(x) = \frac{1}{\Gamma(\delta)} \prod_{i=1}^m \frac{\Gamma(v_i)}{\Gamma(u_i)} (1-x)^{\delta-1} \left\{ 1 + \sum_{n=1}^{\infty} c_n (1-x)^n \right\}, 0 < x \leq 1.$$

The use of the inverse Mellin transform to derive the series expansion in (1.3) is sketched in Section 2. The differential equation initial value problem for the density is described in Section 3, and the recurrence for (1.4) is derived in Section 4.

The examples in Section 5 include the relatively straightforward situation  $m = 2$  and the density of the determinant of a random  $4 \times 4$  positive-definite matrix of trace one, where  $m = 3$ .

## 2. The inverse Mellin transform

The Mellin transform of the density  $f$  is defined by

$$Mf(p) = \int_0^1 x^{p-1} f(x) dx.$$

This is an analytic function in  $\{p : \operatorname{Re} p > 0\}$  and agrees with the meromorphic function

$$p \mapsto \prod_{j=1}^m \frac{\Gamma(v_j) \Gamma(u_j + p - 1)}{\Gamma(u_j) \Gamma(v_j + p - 1)}$$

at  $p = 1, 2, 3, \dots$  thus the two functions coincide in the half-plane by Carlson's theorem. The inverse Mellin transform is

$$f(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} Mf(p) x^{-p} dp,$$

for  $\sigma > 0$ ; it turns out the integral can be evaluated by residues (it is of Mellin-Barnes type). For each  $j$  and each  $n = 0, 1, 2, \dots$  there is a pole of  $Mf(p)$  at

$p = 1 - n - u_j$ ; the hypothesis  $u_i - u_j \notin \mathbb{Z}$  for each  $i \neq j$  implies that each pole is simple. The residue at  $p = 1 - n - u_k$  equals

$$x^{u_k-1+n} \prod_{j=1}^m \frac{\Gamma(v_j)}{\Gamma(u_j) \Gamma(v_j - u_k - n)} \prod_{i \neq k} \Gamma(u_i - u_k - n) \\ \times \lim_{p \rightarrow 1-n-u_k} (p-1+n+u_k) \Gamma(u_k + p - 1).$$

To simplify this we use

$$\Gamma(a-n) = \frac{\Gamma(a)}{(a-n)_n} = (-1)^n \frac{\Gamma(a)}{(1-a)_n}, \\ \lim_{p \rightarrow -p_0} (p+p_0) \Gamma(p_0+p-n) = \lim_{p \rightarrow -p_0} (p+p_0) \frac{\Gamma(p+p_0+1)}{(p+p_0-n)_{n+1}} \\ = \frac{\Gamma(1)}{(-n)_n} = \frac{(-1)^n}{n!}.$$

Thus

(2.1)

$$f(x) = \left( \prod_{i=1}^m \frac{\Gamma(v_i)}{\Gamma(u_i)} \right) \sum_{k=1}^m \frac{x^{u_k-1}}{\Gamma(v_k - u_k)} \prod_{j \neq k} \frac{\Gamma(u_j - u_k)}{\Gamma(v_j - u_k)} \sum_{n=0}^{\infty} \prod_{i=1}^m \frac{(1+u_k-v_i)_n}{(1+u_k-u_i)_n} x^n,$$

(note  $(1+u_k-u_k)_n = n!$ ); in fact this is a Meijer G-function (see [4, 16.17.2]).

### 3. The differential equation

The equation is of Mellin-Barnes type: let  $\partial_x := \frac{d}{dx}$ ,  $D := x\partial_x$  and define the differential operator

$$T(u, v) = -x \prod_{j=1}^m (D+2-v_j) + \prod_{j=1}^m (D+1-u_j).$$

The highest order term is  $(1-x)x^m \partial_x^m$  and the equation has regular singular points at  $x=0$  and  $x=1$ . We find

$$T(u, v) x^c \sum_{n=0}^{\infty} c_n x^n = x^c \sum_{n=0}^{\infty} c_n \left\{ - \prod_{j=1}^m (n+c+2-v_j) x^{n+1} + \prod_{j=1}^m (n+c+1-u_j) x^n \right\} \\ = x^c \sum_{n=1}^{\infty} x^n \left\{ c_n \prod_{j=1}^m (n+c+1-u_j) - c_{n-1} \prod_{j=1}^m (n+c+1-v_j) \right\} \\ + x^c c_0 \prod_{j=1}^m (c+1-u_j).$$

The solutions of the indicial equation are  $c = u_i - 1$ ,  $1 \leq i \leq m$ . Assume  $u_i - u_j \notin \mathbb{Z}$  for  $i \neq j$ . Let  $c = u_1 - 1$  then obtain a solution of  $T(u, v) f(x) = 0$  by solving the recurrence

$$c_n = \prod_{j=1}^m \frac{(u_1 - v_j + n)}{(u_1 - u_j + n)} c_{n-1} = \frac{\prod_{j=1}^m (u_1 - v_j + 1)_n}{n! \prod_{j=2}^m (u_1 - u_j + 1)_n} c_0.$$

Thus the solutions of  $T(u, v)f = 0$  are linear combinations of

$$f_1(x) := x^{u_1-1} {}_mF_{m-1} \left( \begin{matrix} u_1 - v_1 + 1, \dots, u_1 - v_m + 1 \\ u_1 - u_2 + 1, \dots, u_1 - u_m + 1 \end{matrix}; x \right),$$

$$f_i(x) := x^{u_i-1} \sum_{n=0}^{\infty} \prod_{j=1}^m \frac{(u_i - v_j + 1)_n}{(u_i - u_j + 1)_n} x^n,$$

for  $1 \leq i \leq m$  (note the factor  $(u_i - u_i + 1)_n = n!$ ).

LEMMA 1. Suppose  $g$  is differentiable on  $(0, 1]$ ,  $g^{(j)}(1) = 0$  for  $0 \leq j \leq k$  and  $h(x) := (D + s)g(x)$  then  $h^{(j)}(1) = 0$  for  $0 \leq j \leq k - 1$ . Furthermore if  $k \geq 0$  then for  $n \geq 0$

$$\int_0^1 x^n h(x) dx = (s - n - 1) \int_0^1 x^n g(x) dx.$$

PROOF. By induction  $\partial_x^j D = x \partial_x^{j+1} + j \partial_x^j$  for  $j \geq 0$ . Hence  $h^{(j)}(1) = g^{(j+1)}(1) + (j + s)g^{(j)}(1)$ . Next

$$\begin{aligned} \int_0^1 x^n h(x) dx &= s \int_0^1 x^n g(x) dx + \int_0^1 x^{n+1} g'(x) dx \\ &= s \int_0^1 x^n g(x) dx + g(1) - (n + 1) \int_0^1 x^n g(x) dx, \end{aligned}$$

and  $g(1) = 0$  by hypothesis.  $\square$

This is the fundamental initial value system:

$$(3.1) \quad \begin{aligned} T(u, v)f(x) &= 0, \\ f^{(j)}(1) &= 0, 0 \leq j \leq m - 1 \end{aligned}$$

PROPOSITION 1. Suppose  $f$  is a solution defined on  $(0, 1]$  of (3.1) then for  $n \geq 0$

$$\int_0^1 x^n f(x) dx = \prod_{i=1}^m \frac{(u_i)_n}{(v_i)_n} \int_0^1 f(x) dx.$$

PROOF. For  $0 \leq j \leq m$  let  $h_j = \prod_{i=1}^j (D + 1 - u_i)f$ , thus  $h_{j+1} = (D + 1 - u_{j+1})h_j$  and by the Lemma  $h_j^{(k)}(1) = 0$  for  $0 \leq k \leq m - 1 - j$ . Also  $\int_0^1 x^n h_{j+1}(x) dx = -(n + u_{j+1}) \int_0^1 x^n h_j(x) dx$  for  $0 \leq j \leq m - 1$ . By induction  $\int_0^1 x^n h_m(x) dx = (-1)^m \prod_{i=1}^m (n + u_i) \int_0^1 x^n f(x) dx$ .

Similarly  $\int_0^1 x^n x \prod_{i=1}^j (D + 2 - v_i)f(x) dx = (-1)^m \prod_{i=1}^m (n + v_i) \int_0^1 x^{n+1} f(x) dx$ . Thus the integral  $0 = \int_0^1 x^n T(u, v)f(x) dx$  implies the recurrence

$$\int_0^1 x^{n+1} f(x) dx = \prod_{i=1}^m \frac{(u_i + n)}{(v_i + n)} \int_0^1 x^n f(x) dx.$$

Induction completes the proof.  $\square$

Observe that the coefficients  $\{\gamma_i\}$  of the solution  $\sum_{i=1}^m \gamma_i f_i(x)$  of the system are not explicit here, but they are found in the inverse Mellin transform expression.

#### 4. The behavior near $x = 1$ and the recurrence

First we establish the form of the density  $f(x)$  in terms of powers of  $(1-x)$ .

LEMMA 2. For  $\alpha, \beta, \gamma > 0$  and  $0 < x \leq 1$

$$\int_x^1 t^{\alpha-1} (1-t)^{\beta-1} \left(1 - \frac{x}{t}\right)^{\gamma-1} dt = B(\beta, \gamma) (1-x)^{\beta+\gamma-1} {}_2F_1 \left( \begin{matrix} \gamma - \alpha, \beta \\ \beta + \gamma \end{matrix}; 1-x \right).$$

PROOF. Change the variable of integration  $t = 1 - s + sx$  then the integral becomes

$$\begin{aligned} & (1-x)^{\beta+\gamma-1} \int_0^1 (1-s(1-x))^{\alpha-\gamma} s^{\beta-1} (1-s)^{\gamma-1} ds \\ &= (1-x)^{\beta+\gamma-1} \frac{\Gamma(\beta) \Gamma(\gamma)}{\Gamma(\beta+\gamma)} {}_2F_1 \left( \begin{matrix} \gamma - \alpha, \beta \\ \beta + \gamma \end{matrix}; 1-x \right). \end{aligned}$$

This is a standard formula, see [3, (9.1.4), p.239] and is valid in  $0 < x \leq 1$  (where  $|1-x| < 1$ ).  $\square$

Set  $\delta := \sum_{i=1}^m (v_i - u_i)$ .

PROPOSITION 2. There exists a sequence  $\{c_n\}$  such that

$$f(x) = \frac{1}{\Gamma(\delta)} \prod_{i=1}^m \frac{\Gamma(v_i)}{\Gamma(u_i)} (1-x)^{\delta-1} \left\{ 1 + \sum_{n=1}^{\infty} c_n (1-x)^n \right\}.$$

PROOF. Argue by induction. For  $m = 1$  we have (see (1.1))

$$\begin{aligned} f(x) &= \frac{\Gamma(v_1)}{\Gamma(u_1) \Gamma(v_1 - u_1)} x^{u_1-1} (1-x)^{v_1-u_1-1} \\ &= \frac{\Gamma(v_1)}{\Gamma(u_1) \Gamma(v_1 - u_1)} (1-x)^{v_1-u_1-1} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(1-u_1)_n}{n!} (1-x)^n \right\}. \end{aligned}$$

Assume the statement is proven for some  $m \geq 1$ , then  $g = f * h(u_{m+1}, v_{m+1} - u_{m+1}; \cdot)$  has the moments  $\prod_{i=1}^{m+1} \frac{(u_i)_n}{(v_i)_n}$ . The convolution integral (see (1.2)) is a sum of terms

$$\begin{aligned} & C_n \int_x^1 (1-t)^{\delta+n-1} \left(\frac{x}{t}\right)^{u_{m+1}-1} \left(1 - \frac{x}{t}\right)^{v_{m+1}-u_{m+1}-1} \frac{dt}{t} \\ &= C_n x^{u_{m+1}-1} (1-x)^{\delta+n+v_{m+1}-u_{m+1}-1} \frac{\Gamma(\delta+n) \Gamma(v_{m+1}-u_{m+1})}{\Gamma(\delta+n+v_{m+1}-u_{m+1})} \\ &\quad \times {}_2F_1 \left( \begin{matrix} \delta+n, v_{m+1}-1 \\ \delta+v_{m+1}-u_{m+1}+n \end{matrix}; 1-x \right) \end{aligned}$$

by Lemma 2 ; and  $x^{u_{m+1}-1} = 1 + \sum_{j=1}^{\infty} \frac{(1-u_{m+1})_j}{j!} (1-x)^j$ . Thus the lowest power of  $(1-x)$  appearing in  $g$  is  $\delta + v_{m+1} - u_{m+1} - 1$  which occurs for  $n = 0$ . By the inductive hypothesis

$$C_0 = \frac{1}{\Gamma(\delta)} \prod_{i=1}^m \frac{\Gamma(v_i)}{\Gamma(u_i)} \frac{\Gamma(v_{m+1})}{\Gamma(u_{m+1}) \Gamma(v_{m+1} - u_{m+1})},$$

and so the coefficient of  $(1-x)^{\delta+v_{m+1}-u_{m+1}-1}$  in  $g$  is

$$C_0 \frac{\Gamma(\delta) \Gamma(v_{m+1} - u_{m+1})}{\Gamma(\delta + v_{m+1} - u_{m+1})} = \frac{1}{\Gamma(\delta + v_{m+1} - u_{m+1})} \prod_{i=1}^{m+1} \frac{\Gamma(v_i)}{\Gamma(u_i)};$$

this completes the induction.  $\square$

For the next step we need to express  $T(u, v)$  in the form  $x^m (1-x) \partial_x^m + \sum_{j=0}^{m-1} x^j (a_j - b_j x) \partial_x^j$ . Recall the elementary symmetric polynomials in the variables  $\{z_1, \dots, z_m\}$  given by the generating function

$$\prod_{j=1}^m (q + z_j) = \sum_{n=0}^m e_n(z) q^{m-n},$$

so  $e_0(z) = 1$ ,  $e_1(z) = z_1 + z_2 + \dots + z_m$ ,  $e_2(z) = z_1 z_2 + \dots + z_{m-1} z_m$  and  $e_m(z) = z_1 z_2 \dots z_m$ . Thus

$$\prod_{j=1}^m (D + 1 - u_j) = \sum_{j=0}^m e_{m-j}(1-u) (x \partial_x)^j = \sum_{j=0}^m a_j x^j \partial_x^j.$$

Let  $(x \partial_x)^k = \sum_{i=0}^k A_{k,i} x^i \partial_x^i$  then  $x \partial_x (x \partial_x)^j = \sum_{i=0}^j A_{j,i} (i x^i \partial_x^i + x^{i+1} \partial_x^{i+1})$ , so  $A_{k+1,i} = A_{k,i-1} + i A_{k,i}$ . This recurrence has the boundary values  $A_{0,0} = 1$ ,  $A_{1,0} = 0$ ,  $A_{1,1} = 1$ . The solution consists of the Stirling numbers of the second kind, denoted  $S(k, i)$  (see [4, 26.8.22]). Thus

$$\begin{aligned} \sum_{j=0}^m a_j x^j \partial_x^j &= \sum_{j=0}^m e_{m-j}(1-u) \sum_{i=0}^j S(j, i) x^i \partial_x^i \\ &= \sum_{j=0}^m x^j \partial_x^j \sum_{i=j}^m S(i, j) e_{m-i}(1-u), \\ a_j &= \sum_{i=j}^m S(i, j) e_{m-i}(1-u), 0 \leq j \leq m. \end{aligned}$$

In particular  $a_m = 1$ ,  $a_{m-1} = \binom{m}{2} + e_1(1-u)$ , and  $a_0 = e_m(1-u) = \prod_{j=1}^m (1-u_j)$ . Similarly

$$\begin{aligned} \sum_{j=0}^m b_j x^j \partial_x^j &= \prod_{j=1}^m (D + 2 - v_j) = \sum_{j=0}^m e_{m-j}(2-v) (x \partial_x)^j \\ &= \sum_{j=0}^m x^j \partial_x^j \sum_{i=j}^m S(i, j) e_{m-i}(2-v), \\ b_j &= \sum_{i=j}^m S(i, j) e_{m-i}(2-v), 0 \leq j \leq m. \end{aligned}$$

The differential equation leads to deriving recurrence relations for the coefficients  $\{c_n\}$ . Convert the differential operator  $T(u, v)$  to the coordinate  $t = 1 - x$ ; set  $\partial_t := \frac{d}{dt}$  (so that  $\partial_t = -\partial_x$ ). Write (expanding  $x^j = (1-t)^j$  with the binomial

theorem)

$$\begin{aligned}
 T(u, v) &= -x \sum_{j=0}^m b_j x^j \partial_x^j + \sum_{j=0}^m a_j x^j \partial_x^j \\
 &= -\sum_{j=0}^m \sum_{i=0}^{j+1} \binom{j+1}{i} (-t)^i b_j (-\partial_t)^j + \sum_{j=0}^m \sum_{i=0}^j \binom{j}{i} (-t)^i a_j (-\partial_t)^j \\
 &= \sum_{k=-m}^1 (-1)^k \sum_{j=\max(0, -k)}^m \left\{ \binom{j}{j+k} a_j - \binom{j+1}{j+k} b_j \right\} t^{k+j} \partial_t^j.
 \end{aligned}$$

The highest order term ( $k = 1$ ) is  $\sum_{j=0}^m b_j t^{j+1} \partial_t^j$ . The term with  $k = 0$  is  $\sum_{j=0}^m (a_j - (j+1) b_j) t^j \partial_t^j$ . The two bottom terms ( $k = -m, 1 - m$ ) are

$$\begin{aligned}
 (-1)^m (a_m - b_m) \partial_t^m &= 0, \\
 (-1)^{m-1} (a_{m-1} - b_{m-1} - t \partial_t) \partial_t^{m-1} &= (-1)^{m-1} \left( \sum_{i=1}^m (v_i - u_i) - m - t \partial_t \right) \partial_t^{m-1};
 \end{aligned}$$

the remaining terms ( $-m < -k < 0$ ) are

$$(-1)^k \left( \sum_{j=k}^m \left( \binom{j}{j-k} a_j - \binom{j+1}{j-k} b_j \right) t^{j-k} \partial_t^{j-k} \right) \partial_t^k.$$

Apply  $T(u, v)$  to  $t^\gamma$  (with the aim of finding a solution  $t^c \sum_{n=0}^\infty c_n t^n$  to  $T(u, v) f = 0$ ); note  $\partial_t^j t^\gamma = (-1)^j (-\gamma)_j t^{\gamma-j}$ , then

$$\begin{aligned}
 T(u, v) t^\gamma &= \sum_{k=-1}^{m-1} R_k(\gamma) t^{\gamma-k}, \\
 R_{-1}(\gamma) &= \sum_{j=0}^m b_j (-1)^j (-\gamma)_j, \\
 R_{m-1}(\gamma) &= (-\gamma)_{m-1} \left( \sum_{i=1}^m (v_i - u_i) - \gamma - 1 \right) = (-\gamma)_{m-1} (\delta - 1 - \gamma),
 \end{aligned}$$

and

$$\begin{aligned}
 R_k(\gamma) &= (-\gamma)_k R'_k(\gamma), \\
 R'_k &= \left( \sum_{j=k}^m \left( \binom{j}{j-k} a_j - \binom{j+1}{j-k} b_j \right) (-1)^{j-k} (-\gamma + k)_{j-k} \right), \quad 0 \leq k < m.
 \end{aligned}$$

These sums can be considerably simplified (and the Stirling numbers are not needed). Introduce the difference operator

$$\nabla g(c) = g(c) - g(c-1).$$

This has a convenient action, for  $j \geq 0$  and arbitrary  $k$

$$\begin{aligned}\nabla(k-c)_j &= (k-c)_j - (k+1-c)_j = \{k-c - (k+j-c)\}(k+1-c)_{j-1} \\ &= -j(k+1-c)_{j-1}, \\ \nabla^k(-c)_j &= (-1)^k \frac{j!}{(j-k)!} (-c+k)_{j-k}, k \leq j.\end{aligned}$$

Define the polynomials

$$\begin{aligned}p(c) &= \prod_{i=1}^m (c+1-u_i), q(c) = \prod_{i=1}^m (c+2-v_i) \\ q_1(c) &= (1+c\nabla)q(c) = (1+c)q(c) - cq(c-1).\end{aligned}$$

PROPOSITION 3.  $R_{-1}(\gamma) = q(\gamma)$  and for  $0 \leq k \leq m-1$

$$R'_k(\gamma) = \frac{1}{k!} \nabla^k p(\gamma) - \frac{1}{(k+1)!} \nabla^k q_1(\gamma).$$

PROOF. By construction  $\sum_{j=0}^m a_j t^j \partial_t^j t^\gamma = p(\gamma)$  and  $\sum_{j=0}^m b_j t^j \partial_t^j t^\gamma = q(\gamma)$ . Apply  $\frac{1}{k!} \nabla^k$  to both sides ( $\nabla$  acts on the variable  $\gamma$ ) of

$$p(\gamma) = \sum_{j=0}^m a_j t^j \partial_t^j t^\gamma = \sum_{j=0}^m (-1)^j a_j (-\gamma)_j,$$

to obtain

$$\begin{aligned}\frac{1}{k!} \nabla^k p(\gamma) &= \frac{1}{k!} \sum_{j=k}^m (-1)^{j-k} \frac{j!}{(j-k)!} a_j (-\gamma+k)_{j-k} \\ &= \sum_{j=k}^m (-1)^{j-k} \binom{j}{j-k} a_j (-\gamma+k)_{j-k}.\end{aligned}$$

Also

$$\begin{aligned}q_1(\gamma) &= (1+\gamma\nabla)q(\gamma) = (1+\gamma\nabla) \sum_{j=0}^m b_j t^j \partial_t^j t^\gamma = (1+\gamma\nabla) \sum_{j=0}^m (-1)^j b_j (-\gamma)_j \\ &= \sum_{j=0}^m (-1)^j b_j \left\{ (-\gamma)_j - j\gamma(1-\gamma)_{j-1} \right\} = \sum_{j=0}^m (-1)^j b_j (1+j)(-\gamma)_j.\end{aligned}$$

Apply  $\frac{1}{(k+1)!} \nabla^k$  to both sides to obtain

$$\begin{aligned}\frac{1}{(k+1)!} \nabla^k q_1(\gamma) &= \frac{1}{(k+1)!} \sum_{j=k}^m (-1)^{j-k} b_j (1+j) \frac{j!}{(j-k)!} (-\gamma+k)_{j-k} \\ &= \sum_{j=k}^m (-1)^{j-k} \binom{j+1}{j-k} b_j (-\gamma+k)_{j-k}.\end{aligned}$$

This completes the proof. □



Hence

$$\begin{aligned} T(u, v) t^c \sum_{n=0}^{\infty} c_n t^n &= t^c \sum_{n=0}^{\infty} c_n \sum_{k=-1}^{m-1} R_k(n+c) t^{n-k} \\ &= t^c \sum_{n=1-m}^{\infty} t^n \sum_{k=-1}^{m-1} R_k(n+k+c) c_{n+k}. \end{aligned}$$

The recurrence for the coefficients for  $n \geq m$  is

$$\begin{aligned} R_{m-1}(n+c) c_n &= - \sum_{k=1}^{\min(m,n)} R_{m-1-k}(n-k+c) c_{n-k}, \\ &(-n-c)_{m-1} (\delta-1-n-c) c_n = \\ &- \sum_{k=1}^{\min(m-1,n)} (-n+k-c)_{m-1-k} R'_{m-1-k}(n-k+c) c_{n-k} - q(n-m+c) c_{n-m}, \end{aligned}$$

where  $c_i = 0$  for  $i < 0$ . At  $n = 0$  the equation is  $(-c)_{m-1} (\delta-1-c) c_0 = 0$ . Let  $c = 0$  then for  $0 \leq n \leq m-2$  the equations are

$$(-n)_{m-1} (\delta-1-n) c_n = - \sum_{k=1}^n (-n+k)_{m-1-k} R'_{m-1-k}(n-k) c_{n-k},$$

but  $(-n+k)_{m-1-k} = 0$  for  $m-1-k > n-k$  (and  $0 \leq k \leq n$ ) thus the coefficients  $c_0, c_1, \dots, c_{m-2}$  are arbitrary, providing  $m-1$  linearly independent solutions to  $T(u, v) f = 0$ . The recurrence can be rewritten as

$$\begin{aligned} c_{m-1} &= \frac{-1}{(m-1)! (\delta-m)} \sum_{k=1}^{m-1} (-1)^k (m-1-k)! R'_{m-1-k}(n-k) c_{n-k}, \\ c_n &= \frac{-1}{(\delta-1-n)} \left\{ \sum_{k=1}^{m-1} \frac{1}{(-n)_k} R'_{m-1-k}(n-k) c_{n-k} + \frac{1}{(-n)_{m-1}} q(n-m) c_{n-m} \right\}, n \geq m. \end{aligned}$$

Assume that  $\delta \notin \mathbb{Z}$  to avoid poles. But these are different from the desired solution which has  $c = \delta - 1$  as was shown in Proposition 2. The recurrence behaves better in this case. Indeed

$$\begin{aligned} c_n &= \frac{1}{n} \sum_{k=1}^{\min(m-1,n)} \frac{(-n+k-\delta+1)_{m-1-k}}{(-n-\delta+1)_{m-1}} R'_{m-1-k}(n-k+\delta-1) c_{n-k} \\ &+ \frac{1}{n(-n-\delta+1)_{m-1}} q(n-m+\delta-1) c_{n-m}, \end{aligned}$$

which simplifies to

$$\begin{aligned} (4.1) \quad c_n &= \frac{1}{n} \sum_{k=1}^{\min(m-1,n)} \frac{1}{(-n-\delta+1)_k} R'_{m-1-k}(n-k+\delta-1) c_{n-k} \\ &+ \frac{1}{n(-n-\delta+1)_{m-1}} q(n-m+\delta-1) c_{n-m}. \end{aligned}$$

The term with  $c_{n-m}$  occurs only for  $n \geq m$ . The denominator factors are of the form  $(\delta+n-1)(\delta+n-2)\dots(\delta+n-k)$ . If  $n \geq m$  then the smallest factor is

$\delta + n - m > \delta > 0$ ; otherwise the smallest factor is  $\delta$  (for  $k = n$ ). Hence this solution is well-defined for any  $\delta > 0$ .

**THEOREM 1.** *Suppose  $u_1, \dots, u_m, v_1, \dots, v_m$  satisfy  $v_i > u_i > 0$  for each  $i$  then there is a density function  $f$  on  $[0, 1]$  with moment sequence  $\prod_{i=1}^m \frac{(u_i)_n}{(v_i)_n}$  and*

$$f(x) = \frac{1}{\Gamma(\delta)} \prod_{i=1}^m \frac{\Gamma(v_i)}{\Gamma(u_i)} (1-x)^{\delta-1} \left\{ 1 + \sum_{n=1}^{\infty} c_n (1-x)^n \right\},$$

where the coefficients  $\{c_n\}$  are obtained with the recurrence (4.1) using  $c_0 = 1$ , and  $\delta = \sum_{i=1}^m (v_i - u_i)$ .

**PROOF.** The density exists because it is the distribution of the random variable  $\prod_{i=1}^m X_i$  where the  $X_i$ 's are jointly independent and the moments of  $X_i$  are  $\frac{(u_i)_n}{(v_i)_n}$  for each  $i$ . By Proposition 2  $f$  has the series expansion given in the statement. Let  $g(x)$  be the function given in the statement and suppose for now that  $\delta > m$  then  $g$  is a solution of the differential system (3.1) (because of the factor  $(1-x)^{\delta-1}$ ). By Proposition 1  $g$  has the same moments as  $Cf$  for some constant  $C$ . By Proposition 2  $f$  and  $g$  have the same leading coefficient in their series expansions. Hence  $f = g$ . The coefficients  $c_n$  are analytic in the parameters for the range  $\delta > m$ . Each moment  $\int_0^1 x^n f(x) dx$  is similarly analytic and so the formula is valid for all  $\delta > 0$ , by use of analytic continuation from the range  $\delta > m$ .  $\square$

The coefficients occurring in the recurrence (4.1) are expressions in the parameters  $u, v$ , which can be straightforwardly computed, especially with computer symbolic algebra.

## 5. Examples

**5.1. Density for  $m = 2$ .** For the easy case  $m = 2$  we can directly find the density function, in a slightly different form.

Given  $u_1, u_2, v_1, v_2 > 0$  and  $\delta = v_1 + v_2 - u_1 - u_2 > 0$  set

$$\begin{aligned} g(u, v; x) &= \frac{\Gamma(v_1) \Gamma(v_2)}{\Gamma(u_1) \Gamma(u_2) \Gamma(\delta)} \\ &\quad \times x^{u_2-1} (1-x)^{\delta-1} {}_2F_1 \left( \begin{matrix} v_2 - u_2, v_1 - u_2 \\ \delta \end{matrix}; 1-x \right) \end{aligned}$$

then

$$\int_0^1 x^n g(u, v; x) dx = \frac{(u_1)_n (u_2)_n}{(v_1)_n (v_2)_n}, n = 0, 1, 2, \dots$$

PROOF. Consider

$$\begin{aligned}
 & \int_0^1 x^{n+u_1-1} (1-x)^{\delta-1} {}_2F_1 \left( \begin{matrix} v_2 - u_2, v_1 - u_2 \\ \delta \end{matrix}; 1-x \right) dx \\
 &= \sum_{m=0}^{\infty} \frac{(v_2 - u_2)_m (v_1 - u_2)_m}{m! (\delta)_m} \frac{\Gamma(n+u_1) \Gamma(\delta+m)}{\Gamma(u_1+\delta+m+n)} \\
 &= \frac{\Gamma(u_1) (u_1)_n \Gamma(\delta)}{\Gamma(n+u_1+\delta)} \sum_{m=0}^{\infty} \frac{(v_2 - u_2)_m (v_1 - u_2)_m}{m! (n+u_1+\delta)_m} \\
 &= \frac{\Gamma(u_1) (u_1)_n \Gamma(\delta)}{\Gamma(n+u_1+\delta)} \frac{\Gamma(n+u_1+\delta) \Gamma(n+u_1+\delta-v_2-v_1+2u_2)}{\Gamma(n+u_1+\delta-v_2+u_2) \Gamma(n+u_1+\delta-v_1+u_2)} \\
 &= \frac{\Gamma(u_1) \Gamma(u_2) \Gamma(v_2+v_1-u_1-u_2)}{\Gamma(v_2) \Gamma(v_1)} \frac{(u_1)_n (u_2)_n}{(v_2)_n (v_1)_n},
 \end{aligned}$$

for each  $n$ . □

By using standard transformations we can explain the other formulation for  $g(u, v; x)$  near  $x = 0$ . From [3, p.249, (9.5.7)]

$$\begin{aligned}
 F \left( \begin{matrix} \alpha, \delta \\ \gamma \end{matrix}; 1-x \right) &= \frac{\Gamma(\gamma-\alpha-\delta) \Gamma(\gamma)}{\Gamma(\gamma-\alpha) \Gamma(\gamma-\delta)} F \left( \begin{matrix} \alpha, \delta \\ 1+\alpha+\delta-\gamma \end{matrix}; x \right) \\
 &+ \frac{\Gamma(\alpha+\delta-\gamma) \Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\delta)} x^{\gamma-\alpha-\delta} F \left( \begin{matrix} \gamma-\alpha, \gamma-\delta \\ 1+\gamma-\alpha-\delta \end{matrix}; x \right).
 \end{aligned}$$

applied to  $g(u, v; x)$  (provided  $u_1 - u_2 \notin \mathbb{Z}$ ) we find

$$\begin{aligned}
 (5.1) \quad g(u, v; x) &= \frac{\Gamma(v_2) \Gamma(v_1) \Gamma(u_2 - u_1)}{\Gamma(u_1) \Gamma(u_2) \Gamma(v_2 - u_1) \Gamma(v_1 - u_1)} \\
 &\times x^{u_1-1} (1-x)^{\delta-1} {}_2F_1 \left( \begin{matrix} v_2 - u_2, v_1 - u_2 \\ 1+u_1-u_2 \end{matrix}; x \right) \\
 &+ \frac{\Gamma(v_2) \Gamma(v_1) \Gamma(u_1 - u_2)}{\Gamma(u_1) \Gamma(u_2) \Gamma(v_2 - u_2) \Gamma(v_1 - u_2)} \\
 &\times x^{u_2-1} (1-x)^{\delta-1} {}_2F_1 \left( \begin{matrix} v_2 - u_1, v_1 - u_1 \\ 1+u_2-u_1 \end{matrix}; x \right).
 \end{aligned}$$

This is quite similar to the general formula (2.1), and the following standard transformation explains the difference

$$(5.2) \quad {}_2F_1 \left( \begin{matrix} a, b \\ c \end{matrix}; x \right) = (1-x)^{c-a-b} {}_2F_1 \left( \begin{matrix} c-a, c-b \\ c \end{matrix}; x \right).$$

If  $u_1 - u_2 \in \mathbb{Z}$  then there are terms in  $\log x$ . The relevant formula can be found in [3, p. 257, (9.7.5)]. Suppose  $u_2 = u_1 + n$  and  $n = 0, 1, 2, \dots$ ,  $\delta = v_2 + v_1 - 2u_1 - n$

then

$$\begin{aligned}
g(u, v; x) &= \frac{\Gamma(v_2) \Gamma(v_1)}{\Gamma(u_1) \Gamma(u_1 + n) \Gamma(v_2 - u_1) \Gamma(v_1 - u_1)} (1 - x)^{\delta-1} \\
&\times x^{u_1-1} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} (v_2 - u_1 - n)_k (v_1 - u_1 - n)_k (-x)^k \\
&+ \frac{\Gamma(v_2) \Gamma(v_1)}{\Gamma(u_1) \Gamma(u_1 + n) \Gamma(v_2 - u_1 - n) \Gamma(v_1 - u_1 - n)} (1 - x)^{\delta-1} \\
&\times (-1)^n x^{u_1+n-1} (-\log x) \frac{1}{n!} {}_2F_1 \left( \begin{matrix} v_2 - u_1, v_1 - u_1 \\ n + 1 \end{matrix}; x \right) \\
&+ \frac{(-1)^n \Gamma(v_2) \Gamma(v_1)}{\Gamma(u_1) \Gamma(u_1 + n) \Gamma(v_2 - u_1 - n) \Gamma(v_1 - u_1 - n)} x^{u_1+n-1} (1 - x)^{\delta-1} \\
&\times \sum_{k=0}^{\infty} \frac{(v_2 - u_1)_k (v_1 - u_1)_k}{k! (n+k)!} \\
&\{\psi(k+1) + \psi(n+k+1) - \psi(v_2 - u_1 + k) - \psi(v_1 - u_1 + k)\} x^k.
\end{aligned}$$

If  $u_1 - u_2 \notin \mathbb{Z}$  then near  $x = 0$  the density is  $\sim C_0 x^{u_1-1} + C_1 x^{u_2-1}$ , but if  $u_2 = u_1 + n$  then the density  $\sim C_0 x^{u_1-1} + C_1 x^{u_1+n-1} (-\log x)$ .

**5.2. Example: parametrized family with  $m = 3$ .** Consider the determinant of a random  $4 \times 4$  state, that is, a random (with the Hilbert-Schmidt metric) positive-definite matrix with trace one. The moments can be directly computed for the real and complex cases and incorporated into a family of variables with a parameter. Here the variable is 256 times the determinant (to make the range  $[0, 1]$ ) and  $\alpha = \frac{1}{2}$  for  $\mathbb{R}$ ,  $\alpha = 1$  for  $\mathbb{C}$ , and  $\alpha = 2$  for  $\mathbb{H}$  (the quaternions). This example is one of the motivations for the preparation of this exposition. The problem occurred in Slater's study of the determinant of a partially transposed state in its role as separability criterion [5].

The moment sequence is

$$\frac{(1)_n (\alpha + 1)_n (2\alpha + 1)_n}{(3\alpha + \frac{5}{4})_n (3\alpha + \frac{3}{2})_n (3\alpha + \frac{7}{4})_n}, n = 0, 1, 2, \dots;$$

thus  $\delta = 6\alpha + \frac{3}{2}$ . For generic  $\alpha$  the density is

$$\begin{aligned}
&\frac{3(12\alpha + 1)(6\alpha + 1)(4\alpha + 1)}{64\alpha^2} {}_3F_2 \left( \begin{matrix} \frac{3}{4} - 3\alpha, \frac{1}{2} - 3\alpha, \frac{1}{4} - 3\alpha \\ 1 - \alpha, 1 - 2\alpha \end{matrix}; x \right) \\
&- \frac{\Gamma(6\alpha + \frac{5}{2}) \Gamma(3\alpha + \frac{3}{2}) 2^{10\alpha}}{4\alpha \sin(\pi\alpha) \Gamma(\alpha + 1) \Gamma(8\alpha + 1)} x^\alpha {}_3F_2 \left( \begin{matrix} \frac{3}{4} - 2\alpha, \frac{1}{2} - 2\alpha, \frac{1}{4} - 2\alpha \\ 1 - \alpha, 1 + \alpha \end{matrix}; x \right) \\
&+ \frac{(2\alpha + 1)^2 \pi^3 \Gamma(3\alpha + \frac{5}{2}) \Gamma(6\alpha + \frac{5}{2}) 2^{-8\alpha}}{48 \sin(\pi\alpha) \sin(2\pi\alpha) \Gamma(2\alpha + \frac{1}{2}) \Gamma(\alpha + \frac{3}{2})^3 \Gamma(\alpha + 1)^4} x^{2\alpha} \\
&\times {}_3F_2 \left( \begin{matrix} \frac{3}{4} - \alpha, \frac{1}{2} - \alpha, \frac{1}{4} - \alpha \\ 1 + \alpha, 1 + 2\alpha \end{matrix}; x \right).
\end{aligned}$$

For numeric computation at  $\alpha = \frac{1}{2}, 1, 2$  one can employ interpolation techniques; for example

$$f(\alpha_0; x) = \frac{2}{3} (f(\alpha_0 + h; x) + f(\alpha_0 - h; x)) - \frac{1}{6} (f(\alpha_0 + 2h; x) + f(\alpha_0 - 2h; x)) \\ - \frac{1}{6} \left( \frac{\partial}{\partial \alpha} \right)^4 f(\alpha_0 + \xi h; x) h^4,$$

where  $f(\alpha; x)$  denotes the density for specific  $\alpha$  and the last term is the error (for some  $\xi \in (-2, 2)$ ); thus the perturbed densities can be computed by the general formula.

**5.3. Example: the recurrence for  $m = 4$ .** Given  $u_1, \dots, u_4, v_1, \dots, v_4$  define  $p(c) = \prod_{i=1}^4 (c + 1 - u_i)$ ,  $q(c) = \prod_{i=1}^4 (c + 2 - v_i)$ ,  $q_1(c) = (c + 1)q(c) - cq(c - 1)$ ,  $\delta = \sum_{i=1}^4 (v_i - u_i)$ ,

$$R'_0(\gamma) = p(\gamma) - q_1(\gamma), \\ R'_1(\gamma) = \nabla p(\gamma) - \frac{1}{2} \nabla q_1(\gamma), \\ R'_2(\gamma) = \frac{1}{2} \nabla^2 p(\gamma) - \frac{1}{6} \nabla^2 q_1(\gamma),$$

then set  $c_0 = 1$ ,

$$c_1 = \frac{1}{\delta} R'_2(\delta - 1) c_0, \\ c_2 = \frac{1}{2(\delta + 1)} R'_2(\delta) c_1 - \frac{1}{2\delta(\delta + 1)} R'_1(\delta - 1) c_0, \\ c_3 = \frac{1}{3(\delta + 2)} R'_2(\delta + 1) c_2 - \frac{1}{3(\delta + 1)(\delta + 2)} R'_1(\delta) c_1 \\ + \frac{1}{3\delta(\delta + 1)(\delta + 2)} R'_0(\delta - 1) c_0, \\ c_n = \frac{1}{n(\delta + n - 1)} R'_2(n + \delta - 2) c_{n-1} - \frac{1}{n(\delta + n - 2)_2} R'_1(n + \delta - 3) c_{n-2} \\ + \frac{1}{n(\delta + n - 3)_3} R'_0(n + \delta - 4) c_{n-3} + \frac{1}{n(\delta + n - 3)_3} q(n + \delta - 5) c_{n-4},$$

for  $n \geq 4$ .

**5.4. Example: a Macdonald-Mehta-Selberg integral.** Let  $S$  be the 3-dimensional unit sphere  $\{x \in \mathbb{R}^4 : \sum_{i=1}^4 x_i^2 = 1\}$  with normalized surface measure  $d\omega$ . Consider  $\prod_{1 \leq i < j \leq 4} (x_i - x_j)^2$  as a random variable (that is, evaluated at a  $d\omega$ -random point). Interestingly, the maximum value  $\frac{1}{108}$  is achieved at the 24 points with (permutations of the) coordinates  $\left\{ \pm \frac{1}{6} \sqrt{9 \pm 3\sqrt{6}} \right\}$ , which is the zero-set of the rescaled Hermite polynomial  $H_4(\sqrt{6}t)$ . The Macdonald-Mehta-Selberg integral (see [2, p. 319]) implies (for  $\kappa \geq 0$ )

$$\int_S \prod_{1 \leq i < j \leq 4} |x_i - x_j|^{2\kappa} d\omega(x) = \frac{1}{2^{6\kappa}} \frac{\Gamma(1 + 2\kappa) \Gamma(1 + 3\kappa) \Gamma(1 + 4\kappa)}{\Gamma(2 + 6\kappa) \Gamma(1 + \kappa)^3}.$$

For integer values  $\kappa = n$  the Gamma functions simplify to Pochhammer symbols; then by use of formulas like  $(1)_{4n} = 4^{4n} \left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n (1)_n$  the value becomes

$$\mu_n = \frac{1}{108^n} \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n}{\left(\frac{5}{6}\right)_n (1)_n \left(\frac{7}{6}\right)_n}.$$

Let  $f_D$  denote the density function of  $D = 108 \prod_{1 \leq i < j \leq 4} (x_i - x_j)^2$  (by the general results the range of  $D$  is  $[0, 1]$ ). Applying Theorem 1 we find

$$f_D(x) = \frac{\sqrt{2}}{3\pi} (1-x)^{\frac{1}{2}} \left\{ 1 + \frac{221}{216} (1-x) + \frac{156697}{155520} (1-x)^2 + \frac{232223093}{235146240} (1-x)^3 + \dots \right\}.$$

By formula (1.3)

$$\begin{aligned} f_D(x) = & \gamma_1 x^{-\frac{3}{4}} {}_3F_2 \left( \begin{matrix} \frac{5}{12}, \frac{1}{4}, \frac{1}{12} \\ \frac{3}{4}, \frac{1}{2} \end{matrix}; x \right) + \gamma_2 x^{-\frac{1}{2}} {}_3F_2 \left( \begin{matrix} \frac{2}{3}, \frac{1}{2}, \frac{1}{3} \\ \frac{3}{4}, \frac{5}{4} \end{matrix}; x \right) \\ & + \gamma_3 x^{-\frac{1}{4}} {}_3F_2 \left( \begin{matrix} \frac{3}{4}, \frac{7}{12}, \frac{11}{12} \\ \frac{3}{2}, \frac{5}{4} \end{matrix}; x \right), \end{aligned}$$

where

$$\begin{aligned} \gamma_1 &= \frac{\pi}{3\Gamma\left(\frac{3}{4}\right)^2 \Gamma\left(\frac{7}{12}\right) \Gamma\left(\frac{11}{12}\right)}, \\ \gamma_2 &= -\frac{2\sqrt{3}}{3\pi}, \\ \gamma_3 &= \frac{1}{3\pi^3} \Gamma\left(\frac{3}{4}\right)^2 \Gamma\left(\frac{7}{12}\right) \Gamma\left(\frac{11}{12}\right). \end{aligned}$$

It is straightforward to derive a series for the cumulative distribution function  $F_D(x) = \int_0^x f_D(t) dt$ . Figures 1 and 2 are graphs of  $f_D$  and  $F_D$  respectively (of course there is vertical asymptote for  $f_D$ ). For computations we used terms up to the eighth power, with the series in  $x$  for  $0 < x \leq 0.55$  and the  $(1-x)$  series for  $0.55 < x \leq 1$ . For a better view there is a graph of  $F_D(x)$  for  $0 \leq x \leq 0.04$  in Fig. 3 and of  $1 - F_D(x)$  for  $0.4 \leq x \leq 1$  in Fig. 4.

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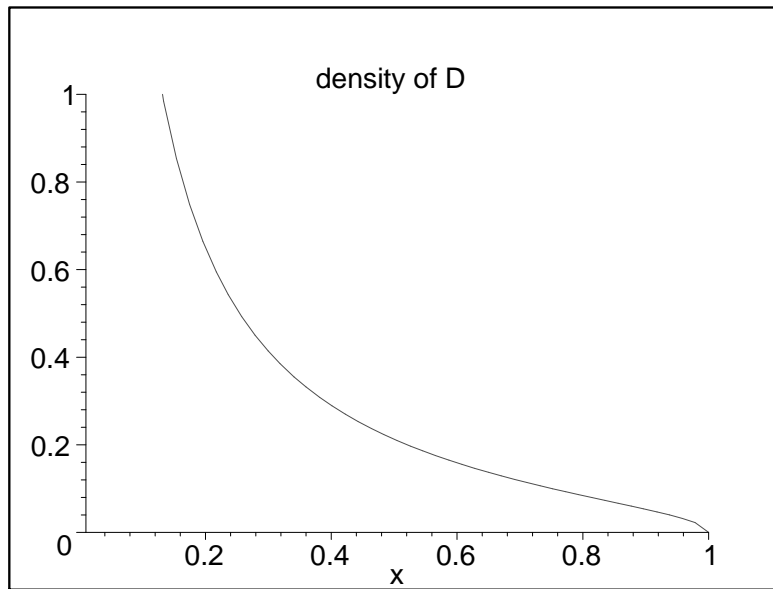


FIGURE 1. Density of D, partial view

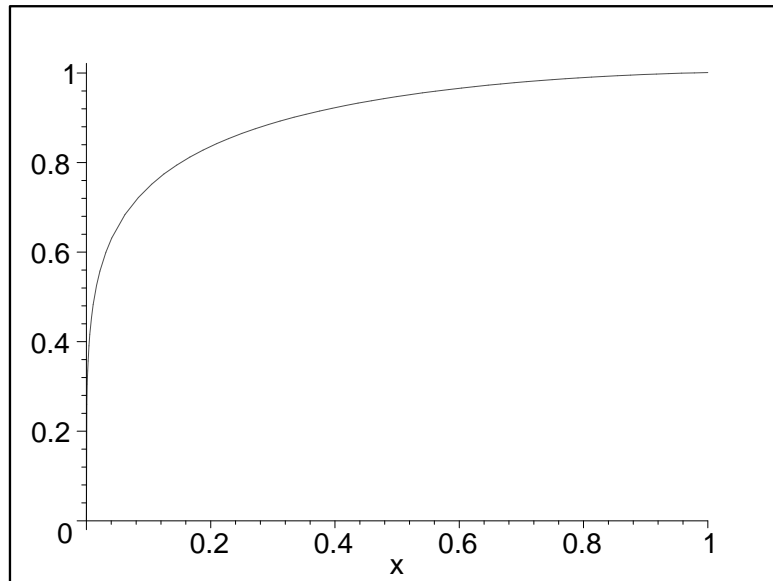
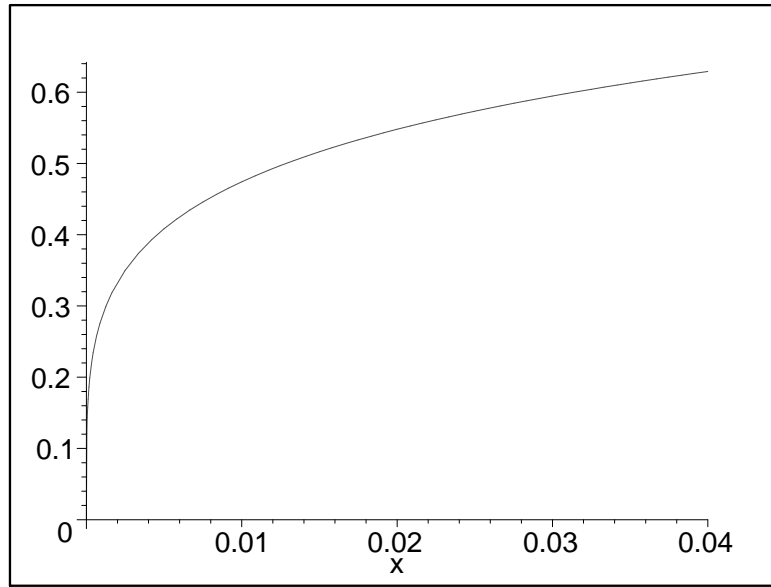
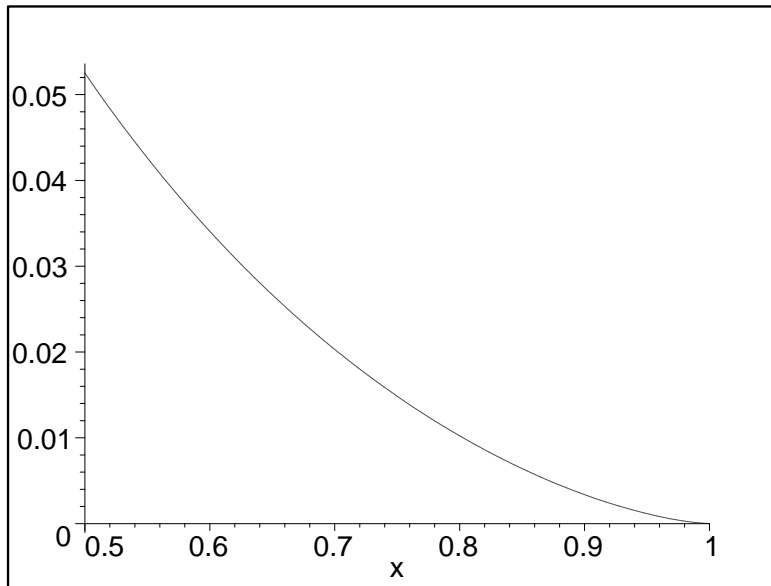


FIGURE 2. Cumulative distribution function of D

FIGURE 3. Part of cumulative distribution of  $D$ FIGURE 4. Part of complementary cumulative distribution of  $D$